

## LINKING AND HOLOMORPHIC HULLS

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### 1. Introduction

If  $X$  and  $Y$  are disjoint compact oriented smooth submanifolds of a smooth oriented manifold  $M$  and are homologous to zero in  $M$ , then the linking number of  $X$  and  $Y$ , denoted  $\text{link}(X, Y)$  (or by  $\text{link}(X, Y; M)$  for clarity) is equal to the intersection number of  $V$  and  $Y$ , where  $(V, X)$  is a compact oriented submanifold with boundary in  $M$ . This can be taken as one of the several equivalent definitions of linking number; here the dimensions  $a, k, m$  of  $X, Y$ , and  $M$  respectively, satisfy  $a+k = m-1$ . We say that  $X$  and  $Y$  are linked if  $\text{link}(X, Y)$  is not zero. Our object is to apply this linking notion of Gauss to the geometry of holomorphic hulls. For example, in the case that the underlying manifold  $M$  is  $\mathbb{C}^n$ , our results say that the polynomially convex hull of one of the sets  $X$  or  $Y$  has a nonempty intersection with the other set, provided that  $X$  and  $Y$  are linked.

Now take  $M$  to be a Stein manifold and let  $X$  be a compact subset of  $M$ . Then the holomorphic hull of  $X$  is

$$\widehat{X} = \{p \in M : |f(p)| \leq \max\{|f(q)| : q \in X\} \text{ for all } f \in A(M)\}$$

where  $A(M)$  is the space of all holomorphic functions on  $M$ .  $\widehat{X}$  is a compact subset of  $M$ . In special cases arising from the maximum principle,  $(\widehat{X}, X)$  is a smooth manifold with boundary which is foliated by complex manifolds with boundaries in  $X$ . In general however,  $\widehat{X}$  is not so nice and may not contain any complex manifolds, or even continuous ones. Nevertheless the perception persists that the pair  $(\widehat{X}, X)$  behaves like a manifold with boundary. This is the motivation for what follows. To adapt the above data on linking to this context we replace  $(V, X)$  with  $(\widehat{X}, X)$  where now  $X$  is an arbitrary compact subset of  $M$ . As before  $Y$  is an oriented manifold disjoint from  $X$  and homologous to zero in  $M$ . Then, when  $X$  and  $Y$  are linked in an appropriate sense, the previous consequence that  $V$  and  $Y$  have a nonzero intersection number will be

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replaced by the cruder statement that  $\widehat{X}$  and  $Y$  have a nonempty intersection. To adapt the hypothesis of the manifolds  $X$  and  $Y$  being linked to the setting in which  $X$  is an arbitrary compact set it suffices to require that  $Y$  not be homologous to zero in  $M \setminus X$ ; when  $X$  is a manifold as above this is equivalent to  $\text{link}(X, Y)$  being nonzero.

**Theorem 1.** *Let  $M$  be a Stein manifold of (complex) dimension  $n$  and  $X$  a compact subset. Let  $Y$  be a compact oriented submanifold of  $M$  of (real) dimension  $k$ , disjoint from  $X$ , and homologous to zero in  $M$ . Suppose that  $Y$  is not homologous to zero in  $M \setminus X$ . Suppose that either*

- (a)  $0 \leq k < n - 1$ , or
- (b)  $k = n - 1$  and  $H^n(M, \mathbb{C}) = 0$ .

*Then  $\widehat{X}$  has a nonempty intersection with  $Y$ .*

**Remarks.** 1. Suppose that  $X$  and  $Y$  are now linked manifolds in  $M$  of dimensions  $a$  and  $k$ , respectively. Then, as  $a + k = 2n - 1$ , the smaller of  $a$  and  $k$  is at most  $n - 1$ . Hence the hull of the set corresponding to the smaller of  $a$  and  $k$  has a nonempty intersection with the other set, unless, in case (b), the smaller is  $n - 1$  and  $H^n(M, \mathbb{C}) \neq 0$ .

2. The cohomology condition in (b) is needed. Consider for  $M$  the product in  $\mathbb{C}^n$  of  $n$  copies of  $\mathbb{C}^*$ , the punctured plane. Let  $X$  be the  $n$ -torus in  $M$ , i.e., the product of  $n$  unit circles. Choose  $Y$  as a  $k = n - 1$  sphere in  $M$  disjoint from  $X$  and such that  $X$  and  $Y$  are linked in  $M$ ; for example,  $Y$  could be a small sphere in the normal space to  $X$  at some point. Then, as  $\widehat{X} = X$ , the intersection of  $\widehat{X}$  and  $Y$  is empty. Of course,  $H^n(M, \mathbb{C}) \neq 0$ .

**Corollary 1.** *Suppose that  $\mathbb{C}^n = S \oplus T$  is an orthogonal decomposition of  $\mathbb{C}^n$  into real linear spaces  $S$  and  $T$  of real dimension  $s$  and  $k$  respectively with  $s > n$  and let  $\pi: \mathbb{C}^n \rightarrow S$  be the orthogonal projection to  $S$ . Let  $E$  be a compact subset of  $S$  and let  $f: E \rightarrow T$  be a continuous map and let  $\text{Gr}(f)$  be the graph of  $f$  in  $\mathbb{C}^n$ . Let  $D$  be a relatively compact component of the complement of  $E$  in  $S$ . Then  $\widehat{\text{Gr}(f)}$ , the polynomially convex hull of  $\text{Gr}(f)$ , covers  $D$ , i.e.,*

$$\pi(\widehat{\text{Gr}(f)}) \supseteq D.$$

The special case of the corollary when  $S$  is complex linear and  $D$  is a ball appeared in [3] with two proofs and a third proof was given by Ahern and Rudin [1]. The second proof in [3], due to J.-P. Rosay, is closest to the methods of this paper. The case  $n = 2$  and  $s = 3$  where  $f$  is a real-valued function on a 2-manifold is of interest. When  $D$  is convex with smooth boundary, a very precise description of the hull is due to Bedford and Klingenberg [7]: the hull is a disjoint union of analytic disks. In other

cases, the structure of the hull is less well understood, as, for example, when  $D$  is a solid torus.

Another phenomenon of linking is the relationship of linking at the boundary of a domain to intersections in the domain. The prototype of such results is the following. Cf. [10, Proposition, p. 383].

**Proposition.** *Let  $(V, X)$  and  $(W, Y)$  be oriented submanifolds with boundary in  $\mathbf{R}^n$  such that  $V$  and  $W$  are contained in the open unit ball  $B$  and such that their boundaries are contained in the unit sphere  $bB$ . Suppose that  $X$  and  $Y$  are disjoint and that  $V$  and  $W$  intersect transversally, if at all. Then*

$$I(V, W) = \text{link}(X, Y; bB).$$

**Remarks.** We are assuming that the linking number is defined. This means that  $\dim(V) + \dim(W) = n$ . Here  $I(V, W)$  denotes the (signed) intersection number of  $V$  and  $W$ . In the case that  $V$  and  $W$  are complex manifolds in  $\mathbf{C}^n$  with their natural orientations, then the intersection number is just the number of points in the intersection. For example, if  $V$  and  $W$  are complex linear spaces of complex dimension  $n$  meeting transversally at the origin in  $\mathbf{C}^{2n}$ , it follows that their boundaries  $X$  and  $Y$ , which are disjoint  $2n - 1$  spheres in the boundary of the unit ball, satisfy  $\text{link}(X, Y; S^{4n-1}) = 1$ . With  $n = 1$ , this fact is used in the standard computations of the Hopf invariant of the Hopf fibration (see [8, pp. 235–239]).

The following is the statement corresponding to the proposition in the case when  $X$  is an arbitrary compact set in a Stein manifold and with  $V$  replaced by a holomorphic hull of  $X$ .

**Theorem 2.** *Let  $M$  be a Stein manifold of complex dimension at least 2, and  $D$  a smoothly bounded relatively compact strictly pseudoconvex domain in  $M$ . Let  $X$  be a compact subset of  $bD$ . Let  $Y$  be a  $k$ -dimensional compact oriented smooth submanifold of  $bD$  with  $0 \leq k \leq n - 2$  which is homologous to zero in  $bD$  and which is disjoint from  $X$ , i.e.,  $Y \subseteq G \equiv bD \setminus X$ , and suppose that there is a  $(k + 1)$ -dimensional submanifold  $W$  of  $D$  such that  $Y = bW$ . Let  $\hat{X}$  be the  $\mathcal{O}_{\bar{D}}$  hull of  $X$ . Suppose that  $Y$  links  $X$  in  $bD$  in the sense that  $Y$  is not homologous to zero in  $G$ . Then  $\hat{X}$  has a nonempty intersection with  $W$ .*

As a consequence we obtain the following corollary originally obtained by the author with E. L. Stout [4] by a different method, extending the Euclidean space case of [2]; also see [6]. The corollary was also proved by Lupaccioulu [9] who obtained more general results related to Theorem 2 in the case of pseudoconcave manifolds. Our approach is perhaps more

geometric. With more elaborate hypotheses, the strict pseudoconvexity of  $D$  in Theorem 2 could be relaxed.

**Corollary 2.** *Let  $M, D, X$  and  $\widehat{X}$  be as in Theorem 2. Each component of  $D \setminus \widehat{X}$  contains in its boundary exactly one component of  $bD \setminus X$ .*

*Proof.* Without loss of generality we can suppose that  $D$  is connected. Then  $bD$  is connected, since  $D$  is Stein and  $n \geq 2$ . It suffices to prove the following. If  $p$  and  $q$  are points in distinct components of  $bD \setminus X$  and if  $W$  is a simple smooth curve in  $D$  joining  $p$  to  $q$ , then  $W$  has a nonempty intersection with  $\widehat{X}$ . Let  $Y$  be  $bW = \{q, -p\}$ , a 0-dimensional submanifold of  $dD$ . The connectedness of  $bD$  implies that  $Y$  is homologous to zero in  $bD$ . Since  $p$  and  $q$  lie in different components of  $bD \setminus X$ ,  $Y$  is not homologous to 0 in  $bD \setminus X$ . Thus we can apply Theorem 2 to conclude that  $\widehat{X}$  meets  $W$ .

## 2. Proof of Corollary 1

Set  $X = \text{Gr}(f)$ . We argue by contradiction and suppose that there exists  $p \in \pi(\widehat{X}) \setminus D$ . Set  $Q = \pi^{-1}(\{p\})$ , a real  $k$ -plane in  $\mathbf{C}^n$ . Then  $\widehat{X} \cap Q$  is empty. Hence  $\widehat{X} \cap Y$  is empty for all geometric  $k$ -spheres  $Y$  in  $\mathbf{C}^n$  of sufficiently large radius  $R$ , which are tangent to  $Q$  at  $(p, 0) \in S \times T = \mathbf{C}^n$ . It is evident and straightforward to check that  $Y$  "links"  $X$ , i.e.,  $Y$  does not bound in  $\mathbf{C}^n \setminus X$ , if  $R$  is sufficiently large. Since  $k = 2n - s < n$ , Theorem 1 implies that  $\widehat{X}$  meets  $Y$ . Contradiction.

## 3. Poincaré duals and linking

We next recall some of the basic facts needed about Poincaré duals and linking. A very nice reference for all of this is the book of Bott and Tu [8]. Our manifolds will be smooth and oriented; for such a manifold  $M$  the  $q$ th de Rham cohomology group will be denoted by  $H^q(M)$ , and the de Rham cohomology with compact support by  $H_c^q(M)$ . For a noncompact oriented manifold  $M$  of dimension  $m$ , Poincaré duality states that

$$H^k(M) = (H_c^{m-k}(M))^*,$$

and also, if  $M$  is of finite type,

$$(H^k(M))^* = H_c^{m-k}(M).$$

If  $Y$  is a closed oriented submanifold of  $M$  of dimension  $k$ , then its Poincaré dual is a closed  $m - k$  form  $\eta_Y$  on  $M$  with the property that

$$(*) \quad \int_Y \alpha = \int_M \alpha \wedge \eta_Y$$

for all closed  $k$  forms  $\alpha$  with compact support in  $M$ . Sometimes to avoid ambiguity we denote the Poincaré dual by  $\eta_Y^M$ . The form is not uniquely determined, but its cohomology class  $[\eta_Y] \in H^{m-k}(M)$  is unique and is also referred to as the Poincaré dual.

Three basic properties of the Poincaré duals are:

(i) Localization. For any tubular neighborhood of  $Y$  in  $M$  there is a Poincaré dual  $\eta_Y$  with support in that neighborhood.

(ii) If the oriented submanifolds  $Y$  and  $W$  of  $M$  meet transversally, then

$$\eta_Y \wedge \eta_W = \eta_{Y \cap W}.$$

(iii) If  $f: M' \rightarrow M$  is an orientation-preserving map, and  $Y$  is an oriented submanifold of  $M$ , then, assuming appropriate transversality,

$$f^*(\eta_Y) = \eta_{f^{-1}(Y)}.$$

In particular, if  $A$  and  $Y$  are oriented submanifolds of  $M$  intersecting transversally, and  $f$  is an inclusion map  $i: A \hookrightarrow M$ , then (iii) gives

$$\eta_Y|_A = i^*(\eta_Y) = \eta_{A \cap Y}^A.$$

Let  $Y$  be a compact oriented submanifold of  $M$ . By localization, we can take  $\eta_Y$  with compact support in  $M$ . We can then ask whether (\*) remains valid if we drop the hypothesis that  $\alpha$  have compact support in  $M$ . By Poincaré duality, this is so, provided that  $M$  has finite type. However, even if  $M$  does not have finite type, we can find a particular  $\eta_Y$  such that

$$(**) \quad \int_Y \alpha = \int_M \alpha \wedge \eta_Y \quad \text{for all closed } k\text{-forms } \alpha \text{ on } M.$$

To see this we choose a tubular neighborhood  $N$  of  $Y$  in  $M$ . Then  $N$  is of finite type and so there is a "compact Poincaré dual" (see [8, p. 51])  $\eta_Y^{iN}$  of  $Y$  in  $N$  such that

$$\int_Y \beta = \int_N \beta \wedge \eta_Y^{iN}$$

for all closed  $k$  forms  $\beta$  in  $N$ ;  $\eta_Y^{iN}$  is a closed  $(m-k)$ -form with compact support in  $N$ . Now define  $\eta_Y$  as the extension to  $M$  of  $\eta_Y^{iN}$  by 0 outside of  $N$ . Then for any closed  $k$ -form  $\alpha$  on  $M$  we have

$$\int_Y \alpha = \int_N \alpha \wedge \eta_Y^{iN} = \int_M \alpha \wedge \eta_Y.$$

Thus (\*\*) holds.

Suppose furthermore that  $Y$  is homologous to zero in  $M$  and let  $\eta_Y$  be chosen so that (\*\*) holds. Then we claim that  $[\eta_Y] = 0$  in  $H_c^{m-k}(M)$ . By Poincaré duality it suffices to show that

$$\int_M \alpha \wedge \eta_Y = 0$$

for all closed forms  $\alpha$  on  $M$ . This follows from (\*\*) because the integral over  $Y$  is zero by Stokes' theorem, since  $Y$  is homologous to zero in  $M$ . Thus there exists a  $(m-k-1)$ -form  $\omega_Y$  with compact support in  $M$  such that  $\eta_Y = d\omega_Y$ .

Suppose that  $X$  and  $Y$  are disjoint oriented compact submanifolds of  $M$ , which are homologous to zero and satisfying  $s+k=m-1$  with dimensions  $s$  and  $k$  respectively. Then  $\text{link}(X, Y)$  is defined and can be computed as follows. Choose  $\eta_X$  and  $\eta_Y$  with compact and disjoint supports. By the last paragraph we have  $\omega_X$  with compact support in  $M$  such that  $d\omega_X = \eta_X$ . Then

$$\text{link}(X, Y) = \int_M \omega_X \wedge \eta_Y.$$

#### 4. Proof of Theorem 1

We argue by contradiction and suppose that  $\widehat{X}$  is disjoint from  $Y$ . Then there exists a relatively compact  $\mathcal{O}_M$ -convex domain  $\Omega$  in  $M$  containing  $\widehat{X}$  and such that  $\overline{\Omega}$  is disjoint from  $Y$ . Let  $\eta_Y$  be a Poincaré dual of  $Y$  in  $M \setminus X$  such that  $\text{spt}(\eta_Y)$  is disjoint from  $\overline{\Omega}$  and (\*\*) holds for  $k$ -forms  $\alpha$  in  $M \setminus X$ . Extending by 0 we can view  $\eta_Y$  as a closed form in  $M$ . As in §3, since  $Y$  is homologous to zero in  $M$ , there exists a  $(2n-k-1)$ -form  $\omega_Y$  with compact support in  $M$  such that  $d\omega_Y = \eta_Y$ . Let  $D_1$  be a relatively compact subdomain in  $M$  containing  $\overline{\Omega} \cup \text{spt}(\omega_Y)$  such that  $bD_1$  is smooth. Choose a relatively compact subdomain  $D_2$  of

$\Omega$  such that  $X$  is contained in  $D_2$  and  $bD_2$  is smooth. Set  $D = D_1 \setminus \overline{D_2}$ . Then  $\text{spt}(\eta_Y) \subseteq D$  and  $bD = bD_1 \cup (-bD_2)$ . As  $Y$  is not homologous to zero in  $M \setminus X$  there exists, by de Rham's theorem, a closed  $k$ -form  $\alpha$  on  $M \setminus X$  such that  $0 \neq \int_Y \alpha$ .

We have

$$\begin{aligned} 0 \neq \int_Y \alpha &= \int_{M \setminus X} \alpha \wedge \eta_Y \quad (\text{by } (**)) \\ &= \int_D \alpha \wedge \eta_Y = \int_D \alpha \wedge d\omega_Y \\ &= (-1)^k \int_D d(\alpha \wedge \omega_Y) = (-1)^k \int_{bD} \alpha \wedge \omega_Y \quad (\text{Stokes}) \\ &= (-1)^k \int_{bD_1} \alpha \wedge \omega_Y + (-1)^k \int_{-bD_2} \alpha \wedge \omega_Y \\ &= (-1)^k \int_{-bD_2} \alpha \wedge \omega_Y \quad (\text{spt}(\omega_Y) \cap bD_1 = \emptyset). \end{aligned}$$

Now in case (a),  $k < n-1$  and so  $2n-k-1 > n$ . Hence  $H^{2n-k-1}(\Omega) = 0$  since  $\Omega$  is Stein [5]. On  $\Omega$ ,  $d\omega_Y = \eta_Y = 0$ . Hence there is a  $(n-k-2)$ -form  $\sigma$  on  $\Omega$  such that  $\omega_Y = d\sigma$  on  $\Omega$ . Thus

$$\int_{-bD_2} \alpha \wedge \omega_Y = \int_{-bD_2} \alpha \wedge d\sigma = (-1)^k \int_{bD_2} d(\alpha \wedge \sigma) = 0$$

by Stokes. This contradicts the choice of  $\alpha$ .

In case (b),  $2n-k-1 = n$ . Since  $(M, \Omega)$  is a Runge pair, it follows from [5] that the natural restriction map  $H^n(M) \rightarrow H^n(\Omega)$  is surjective. As  $H^n(M) = 0$ , we have  $H^n(\Omega) = 0$ , and the argument of case (a) can be applied to arrive at the same contradiction.

### 5. Proof of the Proposition

Extend  $V$  and  $W$  to a neighborhood  $N$  of  $\overline{B}$  and choose Poincaré duals  $\eta_V$  and  $\eta_W$  in  $N$  such that  $\text{spt}(\eta_V) \cap \text{spt}(\eta_W)$  is a compact subset of  $B$ . Then  $\eta_V \wedge \eta_W = \eta_{V \cap W}^B$  is a Poincaré dual of  $V \cap W$ . In particular,  $\int_B \eta_{V \cap W}^B = I(V, W)$ .

Let  $j: bB \rightarrow N$  be the inclusion map. We may assume that  $N$  is a ball. Hence there exists an  $(n-k-1)$ -form  $\omega_Y$  in  $N$  such that  $d\omega_Y = \eta_Y$ . Set  $\eta_X^{bB} = j^*(\eta_V)$  and  $\eta_Y^{bB} = j^*(\eta_W)$ . These are Poincaré duals on  $bB$

with disjoint supports. Set  $\omega_X^{bB} = j^*(\omega_V)$ . Then on  $bB$ , we have

$$d(\omega_X^{bB}) = d(j^*(\omega_V)) = j^*(d\omega_V) = j^*(\eta_V) = \eta_X^{bB}.$$

Thus

$$\begin{aligned} \text{link}(X, Y; bB) &= \int_{bB} \omega_X^{bB} \wedge \eta_Y^{bB} \\ &= \int_{bB} j^*(\omega_V) \wedge j^*(\eta_W) \\ &= \int_{bB} j^*(\omega_V \wedge \eta_W) = \int_{bB} \omega_V \wedge \eta_W \\ &= \int_B d(\omega_V \cap \eta_W) \quad (\text{Stokes}) \\ &= \int_B d\omega_V \wedge \eta_W \quad (\eta_W \text{ is closed}) \\ &= \int_B \eta_V \wedge \eta_W = \int_B \eta_{V \cap W} \\ &= I(V, W). \end{aligned}$$

## 6. Proof of Theorem 2

By replacing  $M$  by an appropriate Stein neighborhood of  $\bar{D}$  in  $M$  we can assume that  $(M, D)$  is a Runge pair, that  $\hat{X}$  is the  $\mathcal{O}_M$ -convex hull of  $X$  and that  $W$  extends to be a submanifold of  $M$  which intersects  $bD$  transversally in  $Y$ .

We argue by contradiction and suppose that  $\hat{X}$  is disjoint from  $W$ . Then there is a relatively compact  $\mathcal{O}_M$  convex domain  $\Omega$  containing  $\hat{X}$  such that  $\bar{\Omega}$  is disjoint from  $W$ . Let  $\eta_W$  be a Poincaré dual on  $M$  with support disjoint from  $\bar{\Omega}$ . Since  $2n - k - 1 > n$  and  $M$  is Stein,  $H^{2n-k-1}(M) = 0$ . Hence there exists a  $(2n - k - 2)$ -form  $\omega_W$  on  $M$  such that  $d\omega_W = \eta_W$ . ( $\eta_W$  is a closed  $(2n - (k + 1))$ -form on  $M$ .)

Let  $j: G \rightarrow M$  be the inclusion map. Set  $\eta_Y^G = j^*(\eta_W)$  and  $\omega_Y^G = j^*(\omega_W)$ . Then  $\eta_Y^G$  is a Poincaré of  $Y$  in  $G$  with compact support in  $G$  such that  $(**)$  holds on  $G$ , at least if we choose the support of  $\eta_W$  close to  $Y$ .

As  $Y$  does not bound in  $G$  there exists a closed  $k$ -form  $\alpha$  on  $G$  such that  $\int_Y \alpha \neq 0$ , by de Rham.

Choose a relatively compact domain  $E_1$  of  $\Omega \cap bD$  such that  $bE_1$  is smooth and  $X \subseteq E_1$ . Set  $E = bD \setminus E_1$ . Then  $Y \subseteq E \subseteq G$  and



$bE = -bE_1 \subseteq \Omega \cap G$ . Thus we have

$$\begin{aligned} 0 \neq \int_Y \alpha &= \int_E \alpha \wedge \eta_Y^G \quad (\text{by } (**); \text{ spt}(\eta_Y^G) \subseteq E) \\ &= \int_E \alpha \wedge d\omega_Y^G = (-1)^k \int_E d(\alpha \wedge \omega_Y^G) \\ &= (-1)^k \int_{bE} \alpha \wedge \omega_Y^G \quad (\text{Stokes}). \end{aligned}$$

We now consider two cases. First suppose  $k < n - 2$ . Then  $2n - k - 2 > n$  and therefore  $H^{2n-k-2}(\Omega) = 0$ , as  $\Omega$  is Stein. Since  $d\omega_Y = \eta_Y = 0$  on  $\Omega$ , there exists a  $(2n - k - 3)$ -form  $\sigma$  on  $\Omega$  such that  $d\sigma = \omega_Y$  on  $\Omega$ . Set the inclusion map  $i: \Omega \cap bD \rightarrow \Omega$  and set  $\sigma' = i^*(\sigma)$ . Then, on  $bE$ ,  $d\sigma' = i^*(\omega_Y) = \omega_Y^G$  and so

$$\begin{aligned} \int_{bE} \alpha \wedge \omega_Y^G &= \int_{bE} \alpha \wedge d\sigma' \\ &= (-1)^k \int_{bE} d(\alpha \wedge \sigma') = 0 \quad (\text{Stokes}), \end{aligned}$$

this contradicts the choice of  $\alpha$ .

In the second case  $k = n - 2$  and  $2n - k - 2 = n$ . Since  $(M, \Omega)$  is a Runge pair, the natural restriction  $H^n(M) \rightarrow H^n(\Omega)$  is surjective [5]. Since  $\omega_W$  is closed on  $\Omega$ , we conclude there exists a closed  $n$ -form  $\phi$  on  $M$  and an  $(n - 1)$ -form  $\theta$  on  $\Omega$  such that

$$\omega_W = \phi + d\theta$$

on  $\Omega$ . Hence

$$\begin{aligned} \int_{bE} \alpha \wedge \omega_Y^G &= \int_{bE} \alpha \wedge \phi + \int_{bE} \alpha \wedge d\theta \\ &= \int_E d(\alpha \wedge \phi) + (-1)^k \int_{bE} d(\alpha \wedge \theta) \end{aligned}$$

by Stokes' theorem. Again by Stokes the last integral vanishes. Also the integral over  $E$  vanishes since  $\alpha \wedge \phi$  is closed because  $\alpha$  and  $\phi$  are closed (and defined on  $E$ ). This again contradicts the choice of  $\alpha$  and completes the proof.

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